LESLIE D. LEWIS

Irrational Numbers Can "In-Spiral" You

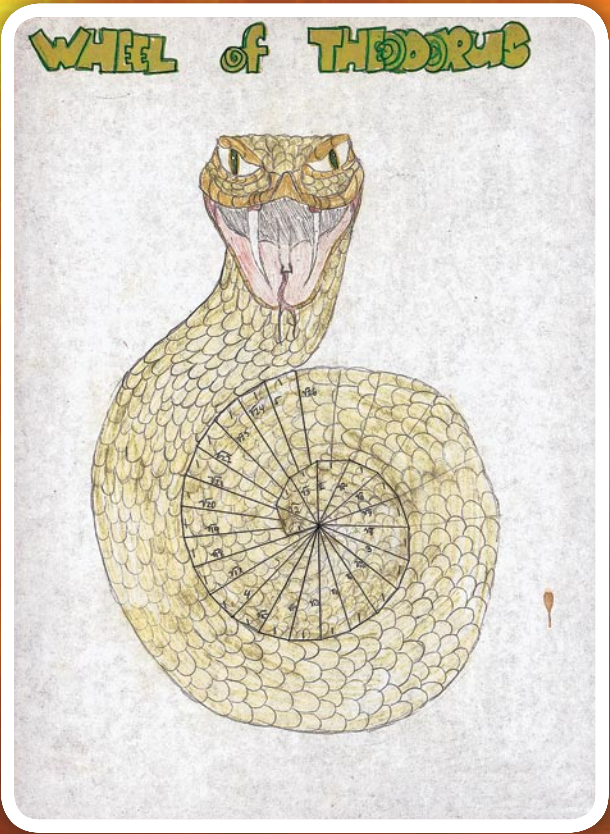
INTRODUCING STUDENTS TO THE PYTHAGOREAN theorem presents a natural context for investigating what irrational numbers are and how they differ from rational numbers. This artistic project allows students to visualize, discuss, and create a product that displays irrational and rational numbers.

Background

WHEN USING THE PYTHAGOREAN THEOREM, STUDENTS find hypotenuse lengths that are not integers. If a student draws a right triangle with legs that have lengths 2 and 3 units, what would he or she expect the length of the hypotenuse to be? For many students, $\sqrt{13}$ is a difficult number to comprehend. It is hoped that students could guess that it must be larger than $\sqrt{9}$ and smaller than $\sqrt{16}$, so it must be between 3 and 4 units long. Experiments with squaring different decimal values do not result in the value of 13. For example, see the list below.

$$3.6^2 = 12.96$$
$$3.7^2 = 13.69$$
$$3.61^2 = 13.0321$$
$$3.60^2 = 12.96$$
$$3.605^2 = 12.996025$$
$$3.606^2 = 13.003236$$

But the process of squaring decimal approximations is an important precursor to understanding irrational numbers.



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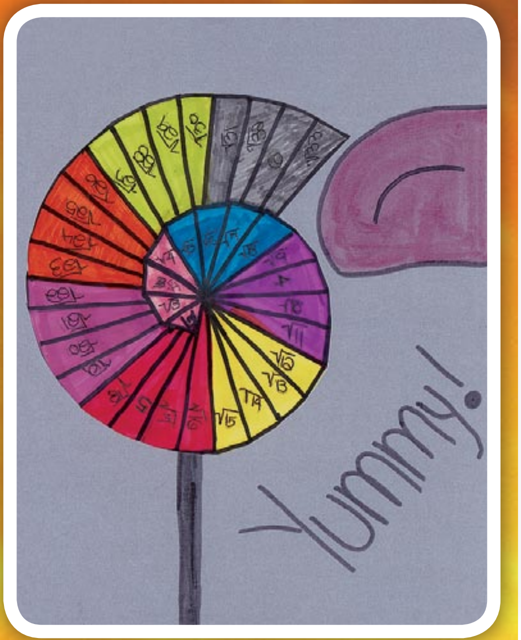

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Somebody in your class will probably type $\sqrt{13}$ into his or her calculator and declare that the square root of 13 is 3.605551275. But square it, you say. When 3.605551275 is squared on the calculator, 13 is shown. But is that accurate? Did the calculator round up or down? Most calculators only display ten digits. Is 3.605551275 only an approximation of the actual root? Square it manually.

$$(3.605551275)^2 = 12.999999996654125$$

Remind students that the numbers that they have seen before either terminate (like 3.605) or have a repetend (like $3.\overline{605}$). These are called *rational numbers* because they can be represented as a fraction with an integer as the numerator and an integer not equal to zero as the denominator. For example,

$$3.605 = \frac{3605}{1000} \text{ and } 3.\overline{605} = \frac{3602}{999}.$$

That

$$3.\overline{605} = \frac{3602}{999}$$

is another interesting topic for discussion. For now, students can verify that 3602/999 is the repeating decimal number $3.\overline{605}$ by manually dividing.

Having explored rational numbers briefly above, explain to students that $\sqrt{13}$ is called an *irrational number*. Its digits do not terminate in zeroes or contain repeating patterns, and it cannot be represented as a fraction of integers. Even though $\sqrt{13}$ and other irrational square roots cannot be written as decimal numbers, they are easy to visualize and represent in a drawing. If a right triangle is drawn and the legs are carefully measured to be 2 inches and 3 inches, then the hypotenuse should be $\sqrt{13}$ inches long. The hypotenuse could be measured or the picture observed to see that the length is between 3 and 4 inches. To the degree of accuracy of a ruler, $\sqrt{13}$ inches is now correctly represented (see **fig. 1**).

Class Activity

THE FUN NOW BEGINS. TO HAVE STUDENTS WORK with this concept in class, they are given a blank piece of paper and a template of a square. On the overhead projector I draw a right triangle, as shown in **figure 2a**. The process is made easier by giving students an accurate right angle. You could use the corner of a 3 × 5 index card and mark off units on the legs of the triangle. (See **fig. 2b**.)

The class and I do the mathematics together. Using the Pythagorean theorem, we calculate the length of the hypotenuse of our isosceles right triangle when the length of each leg is 1 unit.

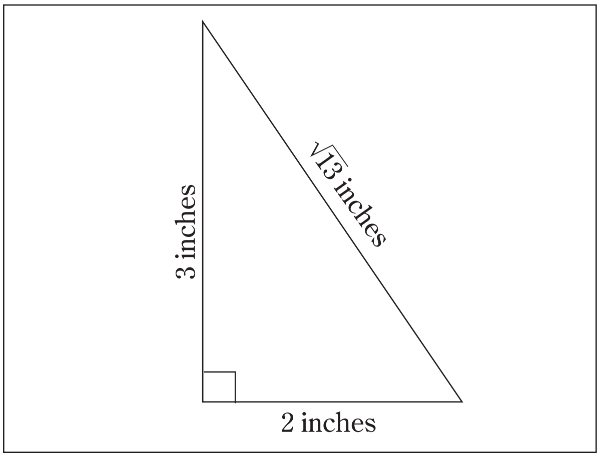

Fig. 1 Visualizing $\sqrt{13}$

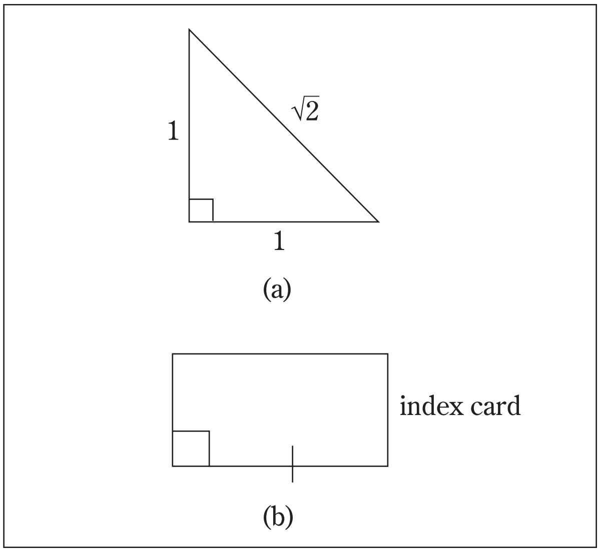

Fig. 2 The (a) drawing is made on the overhead projector; (b) shows that a 3 × 5 card can supply a good right triangle.

$$c^2 = 1^2 + 1^2$$
$$c^2 = 2$$
$$c = \sqrt{2}$$

The students can then see a visual representation of $\sqrt{2}$ units and compare it with a length of 1 unit in the drawing.

To find representations for more irrational lengths, students can now use the calculations and drawing that they have just completed. Using that $\sqrt{2}$ hypotenuse as one leg, they can draw another right triangle with legs 1 and $\sqrt{2}$. Then, using the Pythagorean theorem, each student can calculate that this new hypotenuse must be $\sqrt{3}$ inches long. It is interesting to note that the picture of $\sqrt{3}$ is so much easier to create than the decimal approximation of $\sqrt{3}$ inches.

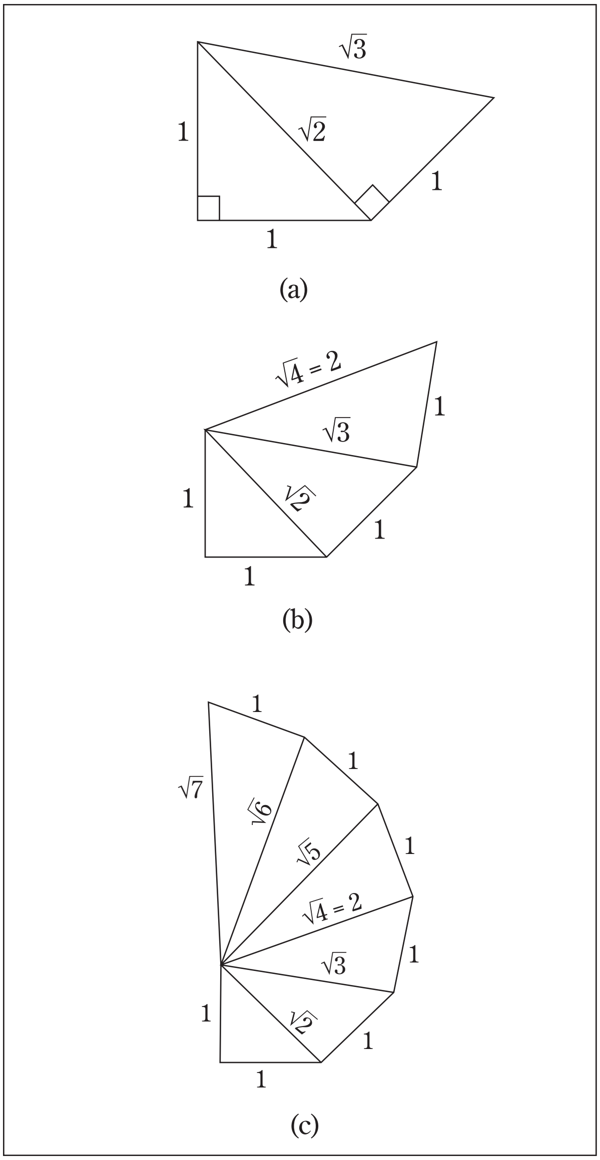

Fig. 3 A spiral progression

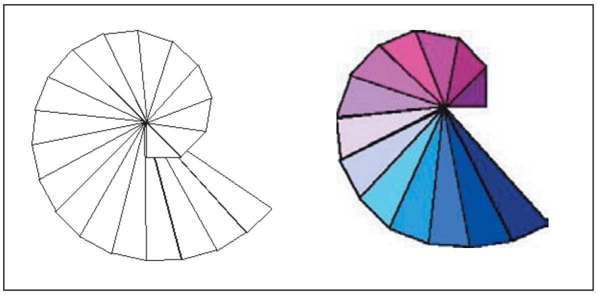

Fig. 4 A completed wheel of Theodorus, with and without color

Continuing in this manner and building on the images as shown in **figures 3a–c**, we can create a graceful spiraling image known as the *wheel of Theodorus* (see **fig. 4**). (Theodorus, a Greek mathematician of the fifth century BC, was a Pythagorean, a member of a group of devoted followers of Pythagoras, and one of Plato's teachers. Little is known about Theodorus; however, Plato gave him credit for proving that the square roots of 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, and 17 are irrational. Many

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people have attempted to determine the technique he used. One popular conjecture involves a spiral, called the wheel of Theodorus, which is composed of contiguous right triangles with hypotenuse lengths equal to the square roots of 2, 3, and 4, up to the square root of 17.)

Asking students to create this spiral by carefully constructing a series of right triangles with one leg remaining 1 unit long and the other leg being the previous hypotenuse is an engaging class exercise. At this point, the classwork ends. The project now becomes an out-of-class task.

Students are given the assignment sheet and the grading rubric shown in the **appendix**. In class, I tell students that their grade will be determined by meeting the criteria of the rubric. Students are required to submit a colorful, labeled drawing with an attached sheet showing their work for the first eight triangles. They are free to create their images with as much or as little detail or artistry as they choose.

Through the years, I have seen projects that are simple but colorful, mathematically correct and incorrect, and delightfully intricate. I have received drawings of sea shells, crustaceans, hair styles, and lollipops (see the artwork featured throughout the article).

Asking students to label calculated side lengths brings to the forefront their understanding of the Pythagorean theorem. They also show their knowledge that some square roots are rational ($\sqrt{4} = 2$), and some are not. They discover that irrational numbers are not very difficult to write in radical form or to express in visual form.

Summary

USING ART PROJECTS IN MY CLASSES HAS GREATLY enriched the endeavor and joy of studying mathematics for me and for my students. Some of the benefits are that students—

- gain a visual understanding of the mathematics involved;
- complete the projects to the degree of patience, artistic ability, and mathematical understanding that they possess;
- who feel daunted by the abstractions of algebra can often thrive using the techniques involved in the art;
- enjoy a break from the paper-and-pencil work that is necessary in mastering mathematics;
- explain the relevance of their art to their families and to the class; and
- who have not previously thrived in mathematics class become leaders during this project.

Adding artistic applications to mathematics classes, exploring the connections between music and the Fibonacci sequence, or building intricate and colorful models of polyhedra both motivate students and increase interest, conversation, pleasure, and—most important—understanding.

My Web site, www2.newton.k12.ma.us/%7Eleslie_lewis/polyhedra/index.htm, shows student work on three-dimensional figures they have made. This exploration helps students become proficient with the intricacies of three-dimensional space and translates wonderfully to network analysis.

Bibliography

Lewis, Leslie. "Polyhedra Creations by Oak Hill 8th Graders." www2.newton.k12.ma.us/%7Eleslie_lewis/polyhedra/index.htm.

Newman, Rochelle, and Martha Boles. *Universal Patterns.* Bradford, MA: Pythagorean Press, 1992.

Pappas, Theoni. *The Joy of Mathematics, Discovering Mathematics All around You.* San Carlos, CA: Wide World Publishing/Tetra House, 1995.

Venters, Diana, and Elaine Krajenke Ellison. *Mathematical Quilts, No Sewing Required.* Berkeley, CA: Key Curriculum Press, 1999.

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Appendix

Wheel of Theodorus Art Project

Assignment

Create a wheel of Theodorus neatly and in color. Mark the unit measures of all of your triangle sides. Feel free to decorate your wheel in a way that demonstrates this spiral in the real world. Attach a lined sheet of paper to your project with your calcuations for the first 8 triangles.

Instructions

1. Using a template for a particular unit length and a right angle, create an isosceles right triangle.
2. Using your template again, add another unit length and right angle to the hypotenuse of your original right triangle.
3. Make a right triangle out of the new unit lengths and the previous hypotenuse.
4. Keep adding a new unit length to the previous hypotenuse at right angles to build new right triangles.
5. When you get to the stage where your right triangles will overlap previous right triangles, draw your hypotenuse *toward the center of the spiral* but do not mark over the previous drawings.
6. Remember to label your figure with all of the dimensions of your successive right triangles. If a hypotenuse has a length that is a rational number, demonstrate that you recognize this fact. (For example, since $\sqrt{4} = 2$, show this on your project.)

Grading rubric

- Include a title for your picture.
- On the front of your picture, include your signature and the date.
- Label all triangle legs and hypotenuses with appropriate lengths.
- Conjoin each new right triangle with the hypotenuse of the previous right triangle.
- Make sure your project is neat.
- Use color unless you mean to emphasize contrast by using black and white.
- Write your labels using radicals unless they can be simplified to rational numbers. For example, you might label a hypotenuse $\sqrt{9} = 3$.
- Connect all of your hypotenuses to the same central point.
- Attach a lined sheet of paper to your art containing your calculations to find lengths of segments (using the Pythagorean theorem) for your first 8 triangles. □