

Drilling Square Holes

A drill bit that produces square holes . . . it defies common sense. How can a revolving edge cut anything but a circular hole? Not only do such drill bits exist, as well as bits for pentagonal, hexagonal, and octagonal holes, but they derive their shape from a simple geometric construction known as a Reuleaux triangle, which was named after Franz Reuleaux, 1829–1905.

To construct a Reuleaux triangle, start with an equilateral triangle of side s (fig. 1). With a radius equal to s and the center at one of the vertices, draw an arc connecting the other two vertices. Similarly, draw arcs connecting the endpoints of the other two sides. The three arcs form the Reuleaux triangle. One of its properties is that of *constant width*, meaning that the figure could be rotated completely between two parallel lines separated by distance s and always be tangent to each.

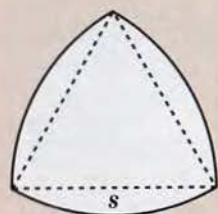


Fig. 1
A Reuleaux triangle

This property of constant width introduced the Reuleaux triangle in a sidebar of our geometry textbook (Moise and Downs 1982, 555). "This figure has constant width," I lectured, "just like a circle." Without thinking, I volunteered, "Imagine it as wheels on a cart." "What sort of cart?" a student asked. "Why, a math cart, to carry my board compass and protractor," I replied, digging myself in deeper. This was the first of several impulsive misstatements I made about the Reuleaux triangle, only to admit my errors after a little reflection. Not in twenty years of teaching had my intuition failed me so completely.

The constant-width property can be used to transport loads, but not by using Reuleaux triangles as wheels. If several logs had congruent Reuleaux triangles as cross sections, bulky items

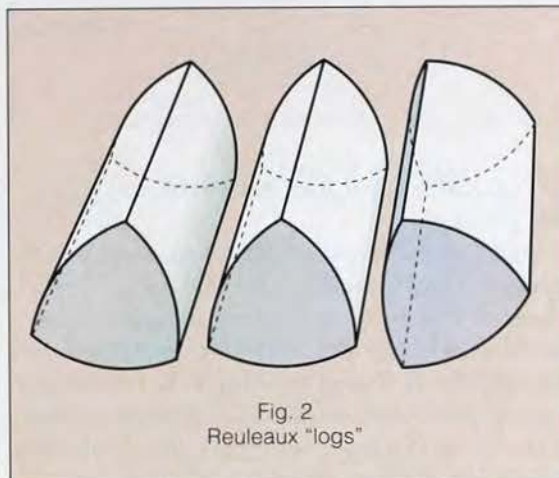


Fig. 2
Reuleaux "logs"

could be rolled on top of a base of them (fig. 2). Movement would occur as logs were transferred from back to front, providing a movable base of constant height. It has been proposed that the Egyptians moved the massive stone blocks for the Great Pyramids in a similar manner.

But the Reuleaux triangle cannot be a wheel. The only conceivable point for the axle, at the center of the triangle, is not the same distance from the Reuleaux triangle's "sides" (fig. 3). If the sides of the equilateral triangle are s , then applying the property that the centroid is two-thirds the distance from a vertex to the opposite sides gives

$$(1) \quad AP = \frac{2}{3} \cdot \frac{s}{2} \sqrt{3} = \frac{\sqrt{3}}{3} s \\ \approx 0.577s,$$

whereas

$$PB = s - \frac{\sqrt{3}}{3} s \\ = s \left(1 - \frac{\sqrt{3}}{3} \right) \\ \approx 0.423s.$$

Even if four Reuleaux-triangle wheels were synchronized, the load would rise and fall continuous-

**What is
the center of
a Reuleaux
triangle?**

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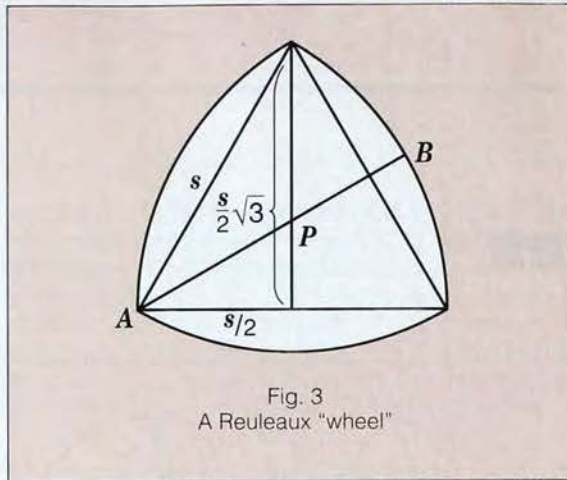


Fig. 3
A Reuleaux "wheel"

ly. You would need to take motion-sickness pills to ride this cart!

"And since it has constant width, it would just fit inside a square whose sides are that width," I continued, trying to regain students' attention. I carefully drew a square circumscribing the Reuleaux triangle (fig. 4). Two of the sides of the triangle are tangent to two sides of the square and two vertices of the triangle intersect the square directly opposite those points of tangency, as shown in figure 4a. If the figure is turned as seen in figure 4b, one side of the Reuleaux triangle is tangent to one side of the square directly opposite one of the vertices of the triangle. All three vertices intersect sides of the square.

"If the Reuleaux triangle just fits inside the square, no matter what position it's in, couldn't it rotate around the inside of the square?" The students needed to be convinced; a model would have to be built. "But if it *did* rotate around the inside, doesn't that mean that a sharp Reuleaux triangle could carve out a square as it rotated?" I had them. "Drill a square hole?" one student countered. "No way!"

That night I cut a Reuleaux triangle with side 10 centimeters from a manila folder to take to class the next day. With a lot of effort, I was able to show that the triangle could rotate around the inside of a ten-centimeter square. "And if this shape was made

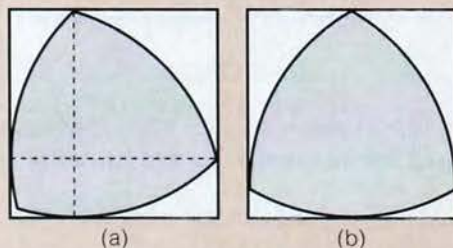


Fig. 4
A Reuleaux triangle inscribed within a square

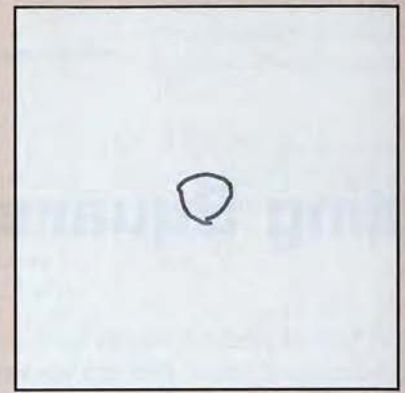


Fig. 5
A tracing of the path of the Reuleaux triangle's centroid

of metal and placed at the end of a rotating shaft, it would cut out a square," I continued, racking up two more falsehoods. First, I implied that the center of the Reuleaux triangle would coincide with the center of a drill's shaft; it cannot. And second, the corners of the holes are not right angles but are slightly rounded.

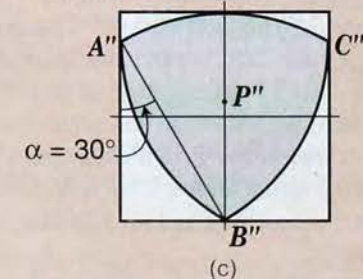
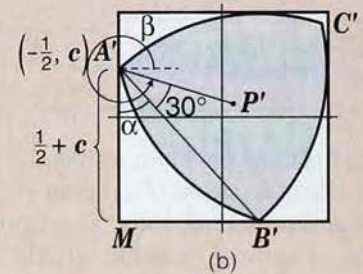
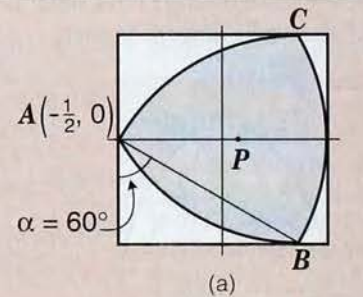


Fig. 6
Reuleaux triangle rotating within a square

How can a Reuleaux bit be used to drill a square hole?

Trying to show that the triangle should be centered at the end of a rotating shaft, I stuck a pen through the triangle's center, and while a student manually rotated the triangle within the square, I traced the center's path on paper (fig. 5). "It's definitely not a single point," I had to admit, holding up the traced curve, "but it sure looks like a circle!" Falsehood 4.

Just what *is* the path of the centroid of a Reuleaux triangle while it is boring a square hole? Assume that the square and the equilateral triangle have sides of length 1. Center the square about the origin and position the Reuleaux triangle so that vertex A is at $(-1/2, 0)$, as in figure 6a. Using (1), the triangle's centroid will be $P(-1/2 + \sqrt{3}/3, 0)$. Next imagine rotating the triangle clockwise through the position in figure 6b, ending up in figure 6c, where the centroid is $P''(0, -1/2 + \sqrt{3}/3)$. The path from P to P'' lies in quadrant I. In figure 6b let α be $m\angle MA'B'$, β be the counterclockwise angle formed by $A'P'$ and a horizontal line through A' , and c be the y-coordinate of point A' . We are interested in the coordinates of P' . Note that $\cos \alpha = 1/2 + c$ and that $\beta = 270^\circ + \alpha + 30^\circ = 300^\circ + \alpha$. Also note that during this rotation from figures 6a through 6b, α goes from 60 degrees to 30 degrees. Since $A'P' = \sqrt{3}/3$, if we measure from the coordinates of $A'(-1/2, c)$, the x and y coordinates of P' can be found:

$$(2) \quad \begin{aligned} x &= \frac{-1}{2} + \frac{\sqrt{3}}{3} \cos(300^\circ + \alpha) \\ &= \frac{-3 + \sqrt{3} \cos \alpha + 3 \sin \alpha}{6}, \end{aligned}$$

and

$$(3) \quad \begin{aligned} y &= c + \frac{\sqrt{3}}{3} \sin(300^\circ + \alpha) \\ &= \left(\cos \alpha - \frac{1}{2} \right) + \frac{\sqrt{3}}{3} \sin(300^\circ + \alpha) \\ &= \frac{-3 + 3 \cos \alpha + \sqrt{3} \sin \alpha}{6} \end{aligned}$$

as α goes from 60 degrees to 30 degrees. Finding the path of the triangle's center in the other three

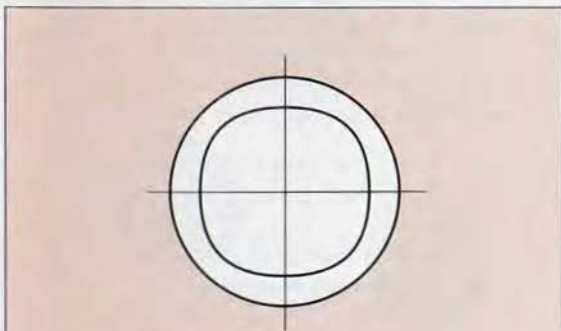


Fig. 7

Path of Reuleaux triangle centroid inside a circle

quadrants is similar in procedure and produces equations that are symmetric to the origin and both axes.

$$\text{Quadrant II:} \quad \begin{aligned} x &= \frac{3 - \sqrt{3} \cos \alpha - 3 \sin \alpha}{6} \\ y &= \frac{-3 + 3 \cos \alpha + \sqrt{3} \sin \alpha}{6} \end{aligned}$$

$$\text{Quadrant III:} \quad \begin{aligned} x &= \frac{3 - \sqrt{3} \cos \alpha - 3 \sin \alpha}{6} \\ y &= \frac{3 - 3 \cos \alpha - \sqrt{3} \sin \alpha}{6} \end{aligned}$$

$$\text{Quadrant IV:} \quad \begin{aligned} x &= \frac{-3 + \sqrt{3} \cos \alpha + 3 \sin \alpha}{6} \\ y &= \frac{3 - 3 \cos \alpha - \sqrt{3} \sin \alpha}{6} \end{aligned}$$

But these equations do *not* describe a circle. In equations (2) and (3), when $\alpha = 30^\circ$, P is on the x-axis at approximately $(0.07735, 0)$. But when $\alpha = 45^\circ$,

$$x = y = \frac{-6 + \sqrt{6} + 3\sqrt{2}}{6},$$

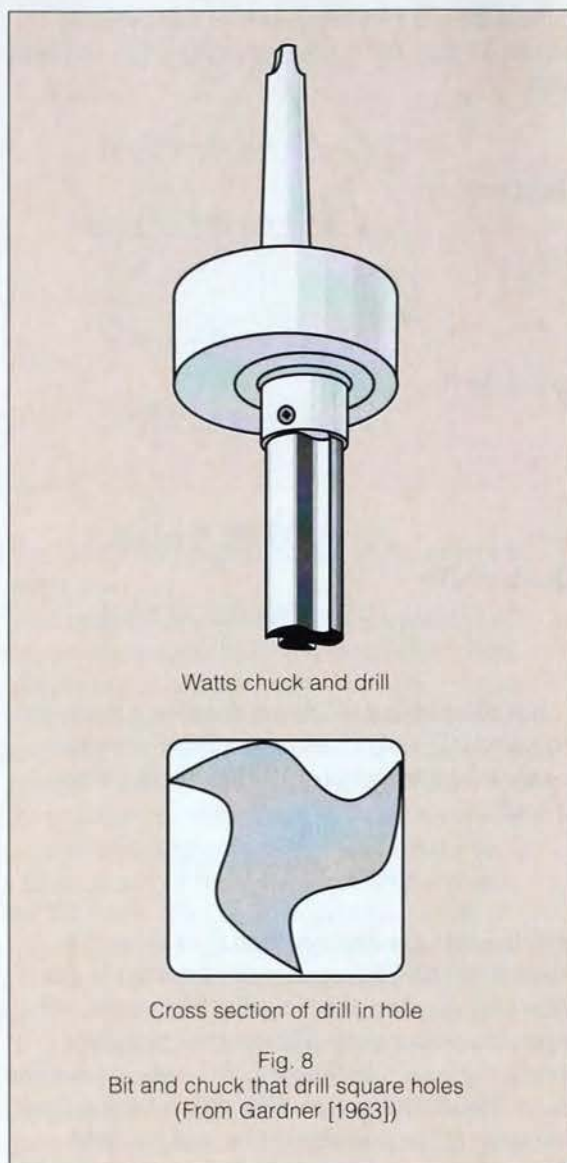
which makes the distance from P' to the origin about 0.08168. This noncircular situation is also shown by graphing the foregoing four parametric equations with a circle whose radius is slightly smaller or larger. In figure 7, the circle is the outer curve. Note that the centroid's path is farther from the circle at the axes than at the midquadrant point.

The Reuleaux triangle's centroid does not follow a circular path. How then is the Reuleaux drill bit contained within the square outline that it is to cut? Harry Watts designed a drill in 1914 with a patented "full floating chuck" to accommodate his irregular bits. Bits for square, pentagonal, hexagonal, and octagonal holes continue to be sold by the Watts Brothers Tool Works in Wilmerding, Pennsylvania. The actual drill bit for the square is a Reuleaux triangle that is concave in three spots to allow for the corners to be cut without shavings obstructing the path (fig. 8).

Even the modified drill bit leaves slightly rounded corners. How rounded? Assume the starting position in figure 9a, in which the Reuleaux triangle is just tangent at point C. As the triangle rotates counterclockwise, C leaves that edge of the square temporarily (labeled C' in fig. 9b) only to rejoin another edge at position C'' in figure 9c. In figure 9b, let α be $m\angle MA'B'$, β be the angle formed by $A'C'$ and the horizontal line through A' , and c be the y-coordinate of A' . Then $\beta = \alpha + 60^\circ - 90^\circ = \alpha - 30^\circ$

Bits exist that can cut pentagonal, hexagonal, and octagonal holes

With this information, the teacher can avoid the blunders I made



and $\cos \alpha = 1/2 + c$. To generate the corner by C , α starts at 30 degrees in **figure 9a** and ends up at 60 degrees in **figure 9c**. From using $A'C' = 1$ and measuring from the coordinates of A' , the coordinates of C' are described by

$$\begin{aligned} x &= \frac{-1}{2} + 1 \cos(\alpha - 30^\circ) \\ &= \frac{-1 + \sqrt{3} \cos \alpha + \sin \alpha}{2}, \end{aligned}$$

and

$$\begin{aligned} y &= c + 1 \sin(\alpha - 30^\circ) \\ &= \left(\cos \alpha - \frac{1}{2} \right) + \sin(\alpha - 30^\circ) \\ &= \frac{-1 + \cos \alpha + \sqrt{3} \sin \alpha}{2}. \end{aligned}$$

The equations for the other three corners are similar and when graphed with the rest of the square yield **figure 10**.

Not only does the Reuleaux triangle have practical and interesting applications and is easy to

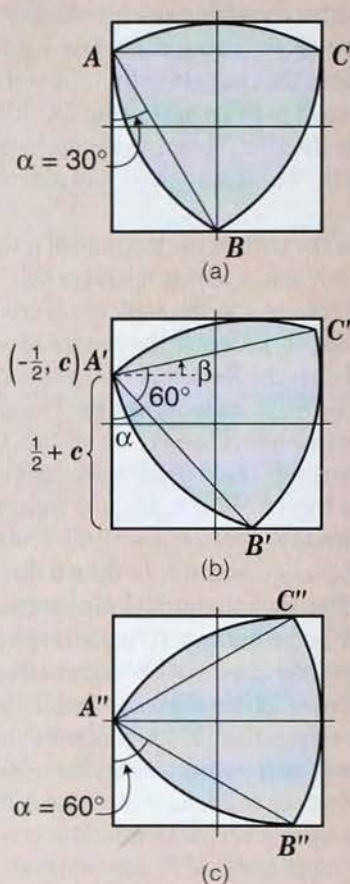


Fig. 9
Reuleaux triangle rotating within a square

describe geometrically, but it generates a lot of discussion owing to its nonintuitive properties. With this background, the teacher can avoid the blunders I made. Further explorations into the topic might include studying other figures of constant width (see Gardner [1963], Rademacher and Toeplitz [1990], and Johnson [1989]); identifying further the curve of the Reuleaux triangle's center as it cuts a square; and noting the shape of bits for pentagonal, hexagonal, and octagonal holes.

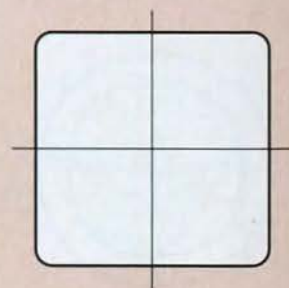



Fig. 10
Path of vertex of Reuleaux triangle
rotating within a square

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